

# Level Repulsion in Constrained Gaussian Random–Matrix Ensembles

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**Abstract.** Introducing sets of constraints, we define new classes of random–matrix ensembles, the constrained Gaussian unitary (CGUE) and the deformed Gaussian unitary (DGUE) ensembles. The latter interpolate between the GUE and the CGUE. We derive a sufficient condition for GUE–type level repulsion to persist in the presence of constraints. For special classes of constraints, we extend this approach to the orthogonal and to the symplectic ensembles. A generalized Fourier theorem relates the spectral properties of the constraining ensembles with those of the constrained ones. We find that in the DGUEs, level repulsion always prevails at a sufficiently short distance and may be lifted only in the limit of strictly enforced constraints.

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## 1. Introduction

Ever since Wigner introduced random-matrix theory (RMT) in nuclear physics in the 1950s [1], that theory has found wide applications in modeling the fluctuation properties of spectra and wave functions of complex systems ranging from vibrating crystals, to microwave resonators, to quantum dots, to atoms, and to the Dirac operator in lattice QCD [2, 3]. Depending on the symmetry of the Hamiltonian, there are three “classical” random-matrix ensembles: the Gaussian orthogonal ensemble (GOE), Gaussian unitary ensemble (GUE), and Gaussian symplectic ensemble (GSE) which consist of real symmetric, complex Hermitean, and complex quaternion matrices, respectively. The corresponding matrix elements are independent, Gaussian-distributed random variables with  $\beta = 1, 2$ , and 4 degrees of freedom, respectively. One of the hallmarks of RMT is level repulsion, i.e., the probability of finding a spacing  $s$  between two closely spaced neighboring levels is proportional to  $s^\beta$ .

Many complex physical systems exhibit spectral fluctuation properties that are in agreement with those of the Gaussian random-matrix ensembles, although the corresponding Hamiltonian matrices have structures that differ considerably from a Gaussian random matrix. Many-body systems with  $k$ -body interactions, for instance, have sparse Hamiltonian matrices, since many-body states that differ in the occupation of more than  $k$  single-particle orbitals cannot be connected by the interaction. Only a few analytical results are known for these  $k$ -body embedded Gaussian ensembles [4], and we refer the reader to the recent reviews [5, 6]. Realistic random-matrix models for nuclei and atoms also have to include spin and isospin symmetry. The resulting two-body random ensemble (TBRE) [7, 8] is mathematically very complicated, and virtually no analytical results are known regarding the fluctuation properties of this important random-matrix model [9]. Another example is given by quasi one-dimensional disordered electronic systems. The corresponding Hamiltonians are band matrices with zero elements outside a small band around the diagonal. The theoretical description of random band matrices is possible due to the simple structure of the Hamiltonian [10].

The embedded Gaussian  $k$ -body ensembles, the TBRE, and the ensembles of random band matrices have one property in common. They can be viewed as *constrained* random-matrix ensembles. Because of the wide occurrence of such matrix ensembles in various branches of theoretical physics, the understanding of their spectral fluctuation properties offers a considerable challenge to theory. So far, the available evidence (mostly based on numerical simulations for matrices of rather small dimensions) points towards fluctuation properties of standard RMT type. The present paper is a first step toward a common theoretical treatment of constrained ensembles. These ensembles are introduced in Section 2, and are defined as Gaussian ensembles of matrices where certain linear combinations of matrix elements vanish. We note that the GOE can ultimately also be viewed as a constrained ensemble; it is obtained from the GUE through the constraint of vanishing imaginary parts of all off-diagonal matrix elements. Remarkably,

some properties of constrained Gaussian random-matrix ensembles do not depend on the details of the constraints but only on their number and symmetry properties (as encoded in systematic degeneracies). In this work, we focus mainly on these rather general properties.

This article is organized as follows. Section 2 introduces constrained Gaussian random matrix ensembles. In Section 3, we employ the Harish–Chandra Itzykson Zuber (HCIZ) integral formula to express the joint probability function for the eigenvalues of the constrained ensemble through an integral over the constraints. In Section 4, we derive a sufficient condition for the existence of quadratic, GUE-like level repulsion. The condition is formulated as a simple inequality that relates the number of constraints and the number of degeneracies to the dimension of Hilbert space. We illustrate this result with a number of examples. In Section 5 we analyse the spectral width and the distribution of matrix element of the CGUE. In Section 6, we show that the joint probability function of the constrained ensemble is related to that of the constraining ensemble by a generalized Fourier transform. In Section 7, we consider *deformed* matrix ensembles. These ensembles interpolate between the GUE and the constrained ensembles and allow us to study situations where the constraints are slowly switched on. We conclude with a Summary.

## 2. Constrained Gaussian unitary random-matrix ensembles

We consider Hermitean matrices acting on a Hilbert space  $\mathcal{H}$  of dimension  $N$ . Together with  $\mathcal{H}$ , we also consider the linear space  $\mathcal{V}$  spanned by the Hermitean matrices acting on  $\mathcal{H}$ : Every linear combination  $aA + bB$  of two such matrices  $A, B$  with real coefficients  $a, b$  is also a Hermitean matrix acting on  $\mathcal{H}$ . In the linear space  $\mathcal{V}$ , we introduce the canonical scalar product in terms of the trace

$$\langle A|B \rangle \equiv \text{Tr}(AB) \quad (1)$$

for every pair  $A, B$  of matrices. This allows us to define an orthonormal basis of  $N^2$  Hermitean basis matrices  $B_\alpha = B_\alpha^\dagger$  in  $\mathcal{V}$  which obey

$$\langle B_\alpha|B_\beta \rangle \equiv \text{Tr}(B_\alpha B_\beta) = \delta_{\alpha\beta} \quad (2)$$

and

$$\sum_{\alpha=1}^{N^2} |B_\alpha \rangle \langle B_\alpha| = \mathbf{1} , \quad (3)$$

where  $\mathbf{1}$  is the unit operator in  $\mathcal{V}$ . Such a set of basis matrices is, for instance, given by matrices that have a unit matrix element somewhere in the main diagonal and zeros everywhere else, or an element  $(1/\sqrt{2})$  somewhere above the main diagonal and its mirror image below the main diagonal and zeros everywhere else, or an element  $(i/\sqrt{2})$  somewhere above the main diagonal and its complex conjugate in the mirrored position below the main diagonal and zeros everywhere else. This choice of a basis is but one example. Any other basis obtained by orthogonal transformations in  $\mathcal{V}$  (with an

orthogonal matrix of dimension  $N^2$ ) from the one just described is equally admissible. In general we have no preference in this respect.

Any Hermitean matrix  $H$  acting on  $\mathcal{H}$  can be expanded in terms of the  $N^2$  Hermitean basis matrices  $B_\alpha$  as

$$H = \sum_{\alpha=1}^{N^2} h_\alpha B_\alpha. \quad (4)$$

Thus,  $H$  can also be viewed as a vector in  $\mathcal{V}$ .

We consider the decomposition of  $\mathcal{V}$  into two orthogonal subspaces labeled  $\mathcal{P}$  and  $\mathcal{Q}$ . The decomposition is defined in terms of orthogonal projection operators

$$\begin{aligned} \mathcal{P} &= \sum_{p=1}^{N_P} |B_p\rangle\langle B_p|, \\ \mathcal{Q} &= \sum_{q=N_P+1}^{N^2} |B_q\rangle\langle B_q|. \end{aligned} \quad (5)$$

We have

$$\mathcal{P}^\dagger = \mathcal{P}, \quad \mathcal{Q}^\dagger = \mathcal{Q}, \quad \mathcal{P}^2 = \mathcal{P}, \quad \mathcal{Q}^2 = \mathcal{Q}, \quad \mathcal{P}\mathcal{Q} = 0, \quad \mathcal{P} + \mathcal{Q} = \mathbf{1}. \quad (6)$$

When we sum over matrices in  $\mathcal{P}$ -space or in  $\mathcal{Q}$ -space, we will sometimes use the summation indices  $p$  and  $q$ , respectively, without referring to the partition explicitly while a summation over all  $N^2$  basis matrices is indicated by Greek summation indices. The operators  $\mathcal{P}$  and  $\mathcal{Q}$  are defined in  $\mathcal{V}$  and have dimension  $N_P$  and  $N_Q = N^2 - N_P$ , respectively. We have been rather explicit in the construction of  $\mathcal{P}$  and  $\mathcal{Q}$  because these operators differ from the projection operators often used in Hilbert space: The latter are defined in terms of the basis vectors spanning Hilbert space, while our projection operators are defined in terms of the basis matrices  $B_\alpha$  and operate in  $\mathcal{V}$ .

In this article, we study constrained ensembles of Gaussian random matrices. Every member  $H$  of the ensemble is constrained to have zero projection onto  $\mathcal{Q}$ -space

$$\mathcal{Q}H = 0 \text{ or } \mathcal{P}H = H. \quad (7)$$

This condition can also be expressed as

$$\langle B_q|H\rangle = 0 \text{ for all } q \text{ or } H = \sum_p B_p \langle B_p|H\rangle. \quad (8)$$

We combine condition (8) with the assumption that the ensemble has a Gaussian distribution. Using the constraint in the form of the first of Eqs. (8), we write the probability density  $\tilde{W}_{\mathcal{P}}(H)$  as

$$\tilde{W}_{\mathcal{P}}(H) d[H] = (2\pi)^{-N_P/2} \exp\left(-\frac{1}{2}\langle H|H\rangle\right) \prod_q \delta(\langle B_q|H\rangle) d[H]. \quad (9)$$

Here  $d[H]$  stands for the product of the differentials of the  $N^2$  independent matrix elements of  $H$ . Eq. (9) defines the constrained Gaussian ensemble of random matrices for the set  $\{B_q\}$  of constraining matrices.

Every unitary transformation  $U$  in Hilbert space induces a transformation  $B_q \rightarrow UB_qU^\dagger$  of the set of matrices defining  $\mathcal{Q}$ -space. We say that the set  $\{B_q\}$  is invariant

under unitary transformations if the vector spaces spanned by the sets  $\{B_q\}$  and  $\{UB_qU^\dagger\}$  are identical. (This is not the case in general). Let us consider an arbitrary unitary transformation  $H \rightarrow UHU^\dagger$  of the matrices in the ensemble (9). All terms on the right-hand side of Eq. (9) remain unchanged except for the arguments of the delta functions. Here, the unitary transformation is tantamount to replacing  $B_q \rightarrow UB_qU^\dagger$ . Therefore,  $\tilde{W}_{\mathcal{P}}$  is unitarily invariant if and only if the set  $\{B_q\}$  has this invariance property. Lack of unitary invariance implies, in general, that the eigenvalues and eigenvectors are correlated random variables.

This is problematic only at first sight. Indeed, the transformation  $\{B_q\} \rightarrow \{UB_qU^\dagger\}$  is exactly compensated by the transformation  $\{H = \sum_p h_p B_p\} \rightarrow \{UHU^\dagger = \sum_p h_p UB_pU^\dagger\}$ . Instead of considering the sets of constraints  $\{B_q\}$  and  $\{UB_qU^\dagger\}$ , we may consider the given set  $\{B_q\}$  and two constrained ensembles, one comprising the matrices  $\{H\}$  as defined by Eq. (9) and the other, all matrices obtained from that set by the operation  $H \rightarrow UHU^\dagger$ , with  $U$  fixed. The matrices  $H$  and  $UHU^\dagger$  possess identical spectra, and in each ensemble the corresponding sets of eigenvalues are encountered with equal probability. Therefore, the two ensembles can differ only in the ordering of the eigenvalues, i.e., by permutations. Such a difference can arise only if the eigenvalue distribution of the original ensemble  $\tilde{W}_{\mathcal{P}}$  is not symmetric.

We group the sets  $\{B_q\}$  of constraining matrices into equivalence classes. Together with a given set  $\{B_q\}$ , each class comprises all sets  $\{UB_qU^\dagger\}$  obtained from  $\{B_q\}$  by arbitrary unitary transformations  $U$  in  $N$  dimensions. The “superensemble”  $W$  is the union of all ensembles  $\tilde{W}$  with constraints in the same equivalence class. By construction,  $W$  is unitarily invariant. We refer to  $W$  as to the constrained Gaussian unitary random-matrix ensemble (CGUE). The CGUE is ergodic if the members of the equivalence class possess symmetric eigenvalue distributions (i.e., distributions that are invariant under all permutations of the eigenvalues). In this case, any result derived for the superensemble also holds for each of its member ensembles. An example is given by the GOE as obtained by constraining the GUE. The GOE is invariant under orthogonal transformations, and the eigenvalue distribution must, therefore, be totally symmetric. Therefore, the superensemble is ergodic. A counterexample is given by the constraint that  $H$  have zero non-diagonal matrix elements in the first row and column. The eigenvalue distribution factorizes and is not symmetric. The constraint is unitarily equivalent, among many others, to the constraint that  $H$  have zero non-diagonal matrix elements in the  $k$ th row and column, with  $k = 2, \dots, N$ . Thus, results derived for the spectral fluctuation properties of the associated CGUE apply jointly to all these constrained ensembles but not to the individual members. We see that our focus on the CGUE prohibits the study of unsymmetrical eigenvalue distribution functions which might be useful, e.g., in studies of symmetry-breaking. It may happen that the vector space spanned by the set  $\{B_q\}$  is invariant under a subgroup of the unitary group. In that case, the equivalence class is generated by the coset space of the unitary group with respect to that subgroup.

For a set of constraints  $\{B_q\}$ , the CGUE is defined by

$$W_{\mathcal{P}}(H)d[H] = (2\pi)^{-N_P/2} \exp\left(-\frac{1}{2}\langle H|H\rangle\right) d[H] \int d[U] \left( \prod_q \delta(\langle UB_q U^\dagger | H \rangle) \right). \quad (10)$$

The integral  $d[U]$  extends over the unitary group in  $N$  dimensions. The ensemble is obviously invariant under unitary transformations of  $H$ . The Haar measure of the unitary group is normalized to one, i.e.

$$\int d[U] = 1. \quad (11)$$

Proceeding as usual, we diagonalize the matrix  $H$  with the help of a unitary matrix  $V$ ,

$$H = VxV^\dagger, \quad (12)$$

where  $x = \text{diag}(x_1, \dots, x_N)$  is the diagonal matrix of the eigenvalues. Here, and in what follows, we denote diagonal matrices by small letters. The integration measure becomes

$$d[H] = \frac{(2\pi)^{N(N-1)/2}}{\prod_{k=1}^N k!} \Delta^2(x) d[x] d[V], \quad (13)$$

where  $\Delta(x)$  denotes the Vandermonde determinant

$$\Delta(x) = \prod_{1 \leq j < k \leq N} (x_k - x_j). \quad (14)$$

Eq. (13) shows that eigenvalues and eigenvectors are uncorrelated random variables. The distribution of the eigenvectors is defined by the Haar measure. Therefore, the joint probability distribution  $P_{\mathcal{P}}(x)$  of the eigenvalues is the object of central interest in this paper. It is given by

$$P_{\mathcal{P}}(x) = \frac{(2\pi)^{(N_Q-N)/2}}{\prod_{k=1}^N k!} \exp\left(-\frac{1}{2}\langle x|x\rangle\right) \Delta^2(x) F_{\mathcal{P}}(x), \quad (15)$$

where

$$F_{\mathcal{P}}(x) \equiv \int d[U] \left( \prod_q \delta(\langle B_q | UxU^\dagger \rangle) \right). \quad (16)$$

This construction of the CGUE may seem unnecessarily involved. Why not start from the constrained ensemble defined in Eq. (9), use the transformation of variables and measures as given by Eq. (13), and integrate  $\tilde{W}$  over the unitary group to obtain the distribution function for the eigenvalues? Following this path, we actually arrive at Eqs. (15) and (16) and, thus, at the CGUE: In this approach, the eigenvalue distributions are always symmetric under permutations of the eigenvalues, since the integration over the unitary group sums over all  $N!$  different unitary matrices that diagonalize a given matrix  $H$  with  $N$  non-degenerate eigenvalues. We believe that our line of reasoning shows more clearly the conceptual framework we use.

It would be desirable to have a classification scheme for the CGUEs. How many such ensembles are there for a given number of constraints? Equivalently, how many different equivalence classes exist for a fixed number of constraining matrices? We have

not investigated these questions yet. In the present paper, we focus attention on joint properties of the CGUEs.

Because of the presence of the function  $F_{\mathcal{P}}(x)$  on the right-hand side of Eq. (15), the spectral statistics of the CGUE differ from those of the GUE. To work out that difference, we must study the function  $F_{\mathcal{P}}(x)$ .

### 3. The HCIZ Integral

To calculate  $F_{\mathcal{P}}(x)$ , we replace the delta functions by Fourier integrals, introducing the set  $\{t_q\}$  of  $N_Q$  integration variables. Then,

$$F_{\mathcal{P}}(x) = (2\pi)^{-N_Q} \int \prod_q dt_q \int d[U] \exp \left( i \sum_q t_q \langle B_q | U x U^\dagger \rangle \right). \quad (17)$$

The integral over the unitary group is the Harish-Chandra Itzykson Zuber (HCIZ) integral [11, 12, 13]. It can be expressed in terms of the eigenvalues  $b_j(\{t_q\})$ ,  $j = 1, \dots, N$  of the matrix

$$B(\{t_q\}) = \sum_q t_q B_q, \quad (18)$$

and is given by

$$\int d[U] \exp \left( i \langle B | U x U^\dagger \rangle \right) = \left( \prod_{j=1}^{N-1} j! \right) i^{-N(N-1)/2} \frac{\det[\exp(ix_k b_l)]}{\Delta(x)\Delta(b)}. \quad (19)$$

The determinant has  $x_k$  in the  $k$ th row, and  $b_l$  in the  $l$ th column, with  $k, l = 1, \dots, N$ . We thus obtain

$$F_{\mathcal{P}}(x) = \frac{\prod_{j=1}^{N-1} j!}{(2\pi)^{N_Q} i^{N(N-1)/2}} \int \prod_q dt_q \frac{\det[\exp(ix_k b_l)]}{\Delta(x)\Delta(b)}. \quad (20)$$

The function  $F_{\mathcal{P}}(x)$  is obviously symmetric with respect to all permutations of the eigenvalues  $x_k$ . The integrand of  $F_{\mathcal{P}}(x)$  in Eq. (20) has that same property with respect to the eigenvalues  $b_l$  of the matrix  $B$ . Both properties are a consequence of the unitary invariance of the CGUE. Multiplying every eigenvalue  $x_k$  by a parameter  $\tau$  so that  $x_k \rightarrow \tau x_k$  for all  $k$ , we get

$$F_{\mathcal{P}}(\tau x) = \tau^{-N_Q} F_{\mathcal{P}}(x). \quad (21)$$

This shows that  $F_{\mathcal{P}}(x)$  is a homogeneous function of the eigenvalues which has a singularity of order  $N_Q$  when all eigenvalues tend to zero simultaneously. However,  $F_{\mathcal{P}}(x)$  does not diverge generically when any one of the eigenvalues vanishes individually.

Using Eq. (20) in Eq. (15), we find for the joint probability distribution of the eigenvalues the expression

$$P_{\mathcal{P}}(x) = \frac{\exp \left( -\frac{1}{2} \langle x | x \rangle \right) \Delta^2(x)}{(2\pi)^{(N+N_Q)/2} i^{N(N-1)/2} N!} \int \prod_q dt_q \frac{\det[\exp(ib_j x_k)]}{\Delta(b)\Delta(x)}. \quad (22)$$

#### 4. Sufficient condition for level repulsion

##### 4.1. Unitary case

We use the function  $F_{\mathcal{P}}(x)$ , as given in Eq. (20), to investigate the spectral statistics and, in particular, level repulsion at short distances for the constrained ensembles. It is obvious that GUE-like level repulsion, as given by the factor  $\Delta^2(x)$  in Eq. (22), will prevail unless the function  $F_{\mathcal{P}}(x)$  has singularities whenever two eigenvalues  $x_j, x_k$  coincide. This prompts us to study especially the singularities of  $F_{\mathcal{P}}(x)$ .

The integrand in Eq. (20) has no singularities: Any zeros of the denominator are canceled by corresponding zeros of the numerator. Indeed, zeros of the denominator arise when two or more eigenvalues  $b_j$  of the matrix  $B$  coincide. We assume that  $b_1 = b_2 = \dots = b_L$  are  $L$  degenerate eigenvalues. Without loss of generality, we may assume that they vanish. (Under translation  $b_j \rightarrow b_j + \Delta b$  with fixed  $\Delta b$  for all  $j$ , the integrand is multiplied by the non-singular function  $\exp(i\Delta b \sum x_k)$ ). Then the integrand becomes

$$\frac{\det[\exp(ib_j x_k)]}{\Delta(x)\Delta(b)} \Big|_{b_1=\dots=b_L=0} = \det \begin{bmatrix} 1 & x_1 & \dots & x_1^{L-1} & e^{ix_1 b_{L+1}} & \dots & e^{ix_1 b_N} \\ 1 & x_2 & \dots & x_2^{L-1} & e^{ix_2 b_{L+1}} & \dots & e^{ix_2 b_N} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 1 & x_N & \dots & x_N^{L-1} & e^{ix_N b_{L+1}} & \dots & e^{ix_N b_N} \end{bmatrix} \frac{\left( \prod_{l=1}^{L-1} \frac{i^l}{l!} \right)}{\left( \prod_{n=L+1}^N b_n \right)^L \Delta(b_{L+1}, \dots, b_N) \Delta(x)}, \quad (23)$$

(where  $\Delta(b_{L+1}, \dots, b_N)$  denotes the Vandermonde determinant of the eigenvalues  $b_{L+1}, \dots, b_N$ ) and is obviously finite. The same conclusion applies when two or more eigenvalues  $x_k$  coincide. For a discussion and alternative calculation of the HCIZ integral in the presence of degeneracies, we refer the reader to Refs. [14, 15].

We conclude that the  $N_Q$ -dimensional integral in Eq. (20) can diverge only because the domain of integration is not bounded. This suggests that we introduce  $N_Q$ -dimensional spherical coordinates  $\prod_q dt_q \equiv t^{N_Q-1} dt d\Omega$  (with  $t$  the radial variable and  $\Omega$  the set of angular variables), and focus attention on the radial integral

$$\int_0^\infty dt t^{N_Q-1} \frac{\det[\exp(ib_j x_k)]}{\Delta(b)}. \quad (24)$$

From Eq. (18), it follows that the eigenvalues are linear homogeneous functions of the variables  $t_q$  so that  $b_j(\{\lambda t_q\}) = \lambda b_j(\{t_q\})$ . This implies  $b_j(t, \Omega) = t b_j(1, \Omega)$ .

We distinguish two cases. (i) There are no *systematic* degeneracies among the eigenvalues  $b_j$ , i.e., there exists no domain of integration with measure  $> 0$  in which eigenvalues are degenerate. (Accidental degeneracies that occur on a set of measure zero of the integration domain are irrelevant.) Then, the Vandermonde determinant in



the denominator yields a factor  $t^{N(N-1)/2}$  and the radial integral becomes

$$\int_0^\infty dt t^{N_Q-1-N(N-1)/2} \det[\exp(itb_j(1, \Omega)x_k)] . \quad (25)$$

The generalization of Eq. (23) to  $L = N$  degenerate eigenvalues shows that the integrand is not singular at  $t = 0$ . The radial integral is sure to converge if  $N_Q < N(N-1)/2$ . (ii) There is a set of  $L$  systematically degenerate eigenvalues. The domain of degeneracy extends out to  $t = \infty$ . In this case, we use the integrand in the form (23). For the radial integral, this yields

$$\int_0^\infty dt t^{N_Q-1+L(L-1)/2-N(N-1)/2} \det \begin{bmatrix} 1 & x_1 & \dots & x_1^{L-1} & e^{ix_1 b_{L+1}} & \dots & e^{ix_1 b_N} \\ 1 & x_2 & \dots & x_2^{L-1} & e^{ix_2 b_{L+1}} & \dots & e^{ix_2 b_N} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 1 & x_N & \dots & x_N^{L-1} & e^{ix_N b_{L+1}} & \dots & e^{ix_N b_N} \end{bmatrix} . \quad (26)$$

Again, the integrand is well behaved at  $t = 0$ , and the integral is guaranteed to converge for  $N_Q + L(L-1)/2 < N(N-1)/2$ . This result can be generalized. Let there be  $J$  sets of  $L_j, j = 1, \dots, J$  degenerate eigenvalues with degeneracy domains that extend out to  $t = \infty$ . Then, quadratic GUE-like level repulsion is guaranteed to occur if the number of constraints  $N_Q$ , the numbers of degenerate levels  $L_j$ , and the matrix dimension  $N$  obey the inequality

$$N_Q + \sum_{j=1}^J L_j(L_j - 1)/2 < N(N-1)/2. \quad (27)$$

The condition (27) is a sufficient (but not necessary) condition for GUE-type level repulsion to occur in the presence of  $N_Q$  constraints. (It may happen that the exponential functions in the integrand provide convergence even if condition (27) is violated.) The inequality (27) makes no reference to the specific structure of the constraints; the only input is the number of constraints and the number of degeneracies. We recall that degeneracies are related to symmetries. In this sense, the number and symmetry properties of the constraints are at the root of the inequality (27).

The possibility to derive the general result (27) without being specific about the nature of the constraints is, of course, due to the unitary symmetry of the CGUE. We have mentioned already that the function  $F_{\mathcal{P}}(x)$  will affect spectral fluctuation properties not only on a very short scale (level repulsion) but also on larger scales. It appears, however, that that influence depends on specific properties of the constraints and requires explicit calculation of  $F_{\mathcal{P}}(x)$ .

#### 4.2. Orthogonal and symplectic case

Instead of considering the Hilbert space of Hermitean matrices, we might have started out from the Hilbert spaces of real symmetric or complex-quaternion matrices. In both cases, we follow the same steps as for the CGUE and arrive at the analogue of Eq. (15)

$$P_{\mathcal{P}}(x) \propto \exp\left(-\frac{1}{2}\langle x|x \rangle\right) |\Delta(x)|^\beta F_{\mathcal{P}}(x) . \quad (28)$$

The factor  $|\Delta(x)|^\beta$  stems from the Jacobian of the eigenvalue/eigenvector transformation for real symmetric ( $\beta = 1$ ), and complex-quaternion matrices ( $\beta = 4$ ). We have omitted the overall normalization constant. Likewise, we have not distinguished the functions  $P_{\mathcal{P}}$  and  $F_{\mathcal{P}}$  from their unitary counterparts as there is no room for confusion. The function  $F_{\mathcal{P}}$  is analogous to Eq. (17) and reads

$$F_{\mathcal{P}}(x) \propto \int \prod_q dt_q \int d[U_\beta] \exp \left( i \sum_q t_q \langle B_q | U_\beta x U_\beta^\dagger \rangle \right). \quad (29)$$

We denote orthogonal and symplectic matrices as  $U_1$  and  $U_4$ , respectively. The integral  $d[U_\beta]$  is over the orthogonal ( $\beta = 1$ ) or symplectic ( $\beta = 4$ ) group in  $N$  dimensions. The constraining matrices  $B_q$  are real and symmetric for  $\beta = 1$  and complex quaternion for  $\beta = 4$ . Eqs. (28) and (29) define the spectral statistics for the constrained Gaussian orthogonal ensemble (CGOE) and the constrained Gaussian symplectic ensemble (CGSE).

Progress is hampered by the fact that the analogue of the HCIZ integral is not available in closed form for the orthogonal and the symplectic groups. However, we might employ an asymptotic expansion of the HCIZ integral that has recently been re-derived in Ref. [16]. We assume that neither the eigenvalues  $x$  nor the eigenvalues of the matrix  $B$  possess any systematic degeneracies. Again, we introduce spherical coordinates  $\prod_q dt_q \equiv t^{N_Q-1} dt d\Omega$ . We are interested in the behavior of the integrand for large values of the radial coordinate  $t$ . Since  $b_i(t, \Omega) = t b_i(1, \Omega)$ , the differences of eigenvalues of  $B$  grow linearly with  $t$ , and we may confine ourselves to the leading term in the asymptotic expansion

$$\int d[U_\beta] \exp \left( i \sum_q t_q \langle B_q | U_\beta x U_\beta^\dagger \rangle \right) \approx \frac{\det[\exp(ib_j x_k)]}{|\Delta(x)\Delta(b)|^{\beta/2}}. \quad (30)$$

Correction terms are inversely proportional to powers of the differences of the eigenvalues ( $b_j - b_k$ ). Following the same arguments as in the unitary case, we find that the radial  $t$ -integration is guaranteed to converge if

$$N_Q < \beta N(N-1)/4. \quad (31)$$

In the absence of any systematic degeneracies of the constraints, the inequality (31) is a sufficient condition for the existence of eigenvalue repulsion of the canonical form  $|x_i - x_j|^\beta$  in the constrained Gaussian ensembles with Dyson index  $\beta$ . The more general result (27) for  $\beta = 2$  and the inequality (31) are central results of our work.

### 4.3. Examples

We present several examples for the CGUE, the CGSE, and the CGOE, which illustrate the power and limitations of the inequalities derived above.

(i) We consider the GOE as obtained by constraining the GUE. The  $\mathcal{P}$ -space consists of real symmetric matrices and the  $\mathcal{Q}$ -space of purely imaginary antisymmetric matrices. We have  $N_Q = N(N-1)/2$  and no systematic degeneracies. The

condition (27) is violated, as must be the case, because the eigenvalue distribution function of the GOE carries  $|\Delta(x)|$  in the first power. This implies that

$$F_{\mathcal{P}}(x) \propto \frac{1}{|\Delta(x)|} . \quad (32)$$

This relation can be proved by employing a relation between complementary distribution functions that is derived in Section 6; see also the Appendix.

(ii) We consider the case where  $\mathcal{Q}$ -space is spanned by the set of diagonal matrices. The number of constraints is  $N$  and the condition (27) is fulfilled. Level repulsion prevails. For  $N \gg 1$ , the corrections to the GUE average level density and the two-point function can be worked out analytically. We use the supersymmetry technique and the standard saddle-point approximation. This approximation consists in neglecting terms which are small of order  $1/N$  and results in an asymptotic expansion in inverse powers of  $N$ . The deviations from the GUE caused by the constraints are of order  $1/N$  and, therefore, do not affect the saddle-point condition. Hence, we find that the corrections to the GUE results vanish like inverse powers of  $N$ .

(iii) This case is converse to the previous one:  $\mathcal{P}$ -space is spanned by the set of diagonal matrices. The number of constraints is  $N_Q = N^2 - N$ , and condition (27) is violated. The levels in  $\mathcal{P}$ -space obey Poisson statistics. Therefore, the function  $F_{\mathcal{P}}$  must compensate the factor  $\Delta^2(x)$  in Eq. (22). Hence we must have

$$F_{\mathcal{P}}(x) \propto \frac{1}{\Delta^2(x)} . \quad (33)$$

It seems difficult to prove the relation (33) by direct calculation.

(iv) Block-diagonal Hermitean matrices: We require that two rectangular blocks in  $H$  carry zero matrix elements. If the two blocks consist of all non-diagonal matrix elements in row  $k$  and in column  $k$ , the number of constraints is  $2(N - 1)$ . In this case,  $N - 2$  of the eigenvalues of  $B$  are degenerate (i.e., vanish), and the remaining two eigenvalues differ only in sign. The condition (27) is violated (albeit weakly). These considerations are easily extended to bigger blocks.

(v) Another example is furnished by the EGUE, the embedded Gaussian unitary ensemble with  $k$ -body interactions [4, 5, 6]. This ensemble differs from a CGUE in that its matrices  $B_p$  do not obey the normalization (2). However, the corresponding CGUE is dense in the EGUE. With  $l$  the number of degenerate single-particle states, and  $m$  the number of identical Fermions, we have  $N = \binom{l}{m}$ . The number of independent  $k$ -body matrices  $B_p$  is  $N_P = 2\left(\binom{l}{k}\right)^2 - \binom{l}{k}$ . According to condition (27), level repulsion is guaranteed if there are no degeneracies and if  $N(N + 1)/2 < N_P$ , or if

$$\frac{1}{2} \binom{l}{m} \left( \binom{l}{m} + 1 \right) < \binom{l}{k} \left( 2 \binom{l}{k} - 1 \right) . \quad (34)$$

For arbitrary  $l$ , this condition is met only for  $k = m$ , and there it is expected. Unfortunately, we cannot draw any non-trivial conclusions for the EGUE.

(vi)  $\mathcal{P}$ -space consists of all traceless Hermitean matrices. The single constraint ( $N_Q = 1$ ) is proportional to the unit matrix, and all  $N$  eigenvalues of the constraint are

degenerate. The probability distribution (22) can be worked out by means of Eq. (23), and the result differs from the GUE eigenvalue distribution only by an overall factor  $\delta(x_1 + x_1 + \cdots + x_N)$ , which yields a GUE spectrum with vanishing centroid. The spectral fluctuations are identical to the GUE though inequality (27) is not fulfilled. This demonstrates that the conditions (27) and, by the same token, (31) are only sufficient: Their fulfillment guarantees level repulsion with power  $\beta$ , and they cannot be fulfilled if there is no level repulsion with power  $\beta$ . However, no conclusion can be drawn about level repulsion if these inequalities are violated.

(vii) We consider the GUE as obtained by constraining the GSE. The number of constraints is  $N_Q = N(N - 1)$ , there are no systematic degeneracies, and the Dyson index is  $\beta = 4$ . The inequality (31) is violated, as it must be.

(viii) We consider an ensemble of real symmetric block-diagonal matrices consisting of two blocks with equal dimensions. Clearly, eigenvalues belonging to different blocks are uncorrelated. The  $\mathcal{Q}$ -space is given by the chiral GOE with quadratic blocks. The number of constraints imposed on the GOE is  $N_Q = N^2/4$ , there are no systematic degeneracies, and  $\beta = 1$ . The inequality (31) is violated, as must be the case.

(ix) We consider the chiral GOE as obtained by constraining the GOE. Matrices of the chiral GOE consist of two rectangular off-diagonal blocks of size  $n \times m$ . For  $m \geq n$ , there are  $m - n$  zero eigenvalues, and the remaining  $2n$  eigenvalues come in  $n$  pairs with opposite signs. The correlation of the positive eigenvalues is determined by the expression [18]

$$\prod_{1 \leq j < k \leq n} |x_k^2 - x_j^2| \prod_{l=1}^n x_l^{|m-n|}. \quad (35)$$

We consider two cases. First, we set  $m = n$ . This case is converse to example (viii), as the roles of  $\mathcal{P}$ -space and  $\mathcal{Q}$ -space are interchanged. According to expression (35), eigenvalues with equal signs repel each other linearly, but there is no repulsion between the smallest positive eigenvalue and the largest negative eigenvalue. The number of constraints is  $N_Q = n(n + 1) = (N/2)(N/2 + 1)$ , there are no systematic degeneracies, and  $\beta = 1$ . The inequality (31) is violated, as it must be. For the second case of the chiral GOE, we set  $m = n + 1$ . In this case, the eigenvalue spectrum has one zero eigenvalue, and  $n$  pairs of eigenvalues with opposite signs. We have  $N_Q = (n + 1)^2 = (N + 1)^2/4$ , there are no systematic degeneracies, and  $\beta = 1$ . The inequality (31) is violated. However, inspection of expression (35) shows that there is linear level repulsion between any pair of neighboring levels. In particular, the zero eigenvalue repels the smallest positive eigenvalue. Again, we cannot draw any conclusion about level repulsion from the inequality (31) since it is violated.

## 5. Further properties of the CGUE

Constraints affect not only the spectral fluctuations of the CGUE but also the spectral width, and the distribution of matrix elements of the CGUE. To show this for the

spectral width, we recall the usual normalization condition  $\overline{H_{\mu\nu}^2} = \lambda^2/N$  of the matrix elements  $H_{\mu\nu}$  of the GUE. Here  $N$  is the matrix dimension, the overbar denotes the ensemble average, and  $2\lambda$  is the radius of Wigner's semicircle. With this normalization, the spectral width of the GUE has the value  $(1/N)\text{Trace}\overline{H^2} = \lambda^2$ . To work out  $(1/N)\text{Trace}\overline{H^2}$  for the CGUE, we start from the normalized distribution function of the eigenvalues,

$$P_{\text{CGUE}}(x) = \mathcal{N} \Delta^2(x) F(x) \exp\left(-\frac{N}{2\lambda^2} \sum_j x_j^2\right). \quad (36)$$

Here  $\mathcal{N}$  is a normalization factor. For  $F(x) = 1$  (no constraints) this function yields the GUE with the above-mentioned normalization. To work out the normalization factor  $\mathcal{N}$ , we relate  $P_{\text{CGUE}}(x)$  to the distribution function  $P_{\mathcal{P}}(x)$  defined in Eq. (15) with normalization factor  $\mathcal{N}_0 = (2\pi)^{(N_Q-N)/2} / \prod k!$ . We substitute  $x_j = y_j \lambda / \sqrt{N}$  and use the fact that  $F(x) = (\sqrt{N}/\lambda)^{N_Q}$  to obtain

$$\mathcal{N} = \left(\frac{N}{\lambda^2}\right)^{(N^2-N_Q)/2} \mathcal{N}_0. \quad (37)$$

A straightforward calculation then yields

$$\left(\frac{1}{N} \text{Trace } \overline{H^2}\right)_{\text{CGUE}} = \lambda^2 \left[1 - \frac{N_Q}{N^2}\right]. \quad (38)$$

This shows that in comparison with the GUE, the spectral width of the CGUE is reduced by the factor  $\sqrt{1 - N_Q/N^2}$ . Every constraint – irrespective of whether it is located on the main diagonal or not, or whether it affects the the real or the imaginary elements of  $H$  – yields the same contribution to the argument of the square root.

For the GUE, the matrix elements are Gaussian-distributed random variables. Because of the constraints, this is not so for the CGUE. To show this, we calculate the variance

$$V_{\text{CGUE}} = \left[\frac{1}{N} \text{Trace } H^2\right]^2 - \left[\frac{1}{N} \text{Trace } \overline{H^2}\right]^2 \quad (39)$$

of  $(1/N)\text{Trace } H^2$  in two ways. Using  $P_{\text{CGUE}}(x)$  as given by Eq. (36) we obtain

$$V_{\text{CGUE}} = \frac{2\lambda^4}{N^2} \left(1 - \frac{N_Q}{N^2}\right). \quad (40)$$

On the other hand, Eq. (38) implies for the CGUE

$$\overline{H_{\mu\nu} H_{\nu\mu}} = \frac{\lambda^2}{N} \left(1 - \frac{N_Q}{N^2}\right). \quad (41)$$

Assuming now that the matrix elements do have a Gaussian distribution, we find for the variance of  $(1/N)\text{Trace } H^2$  the value

$$\frac{2\lambda^4}{N^2} \left(1 - \frac{N_Q}{N^2}\right)^2. \quad (42)$$

This disagrees with Eq. (40) and shows that the deviations from the Gaussian distribution increase monotonically with increasing number of constraints. While the matrix elements of the constrained ensembles defined by Eq. (9) do have a Gaussian distribution, this is not the case for their unitarily invariant counterparts defined in Eqs. (15) and (16).

## 6. Complementary distribution functions

Equation (22) expresses the joint probability function  $P_{\mathcal{P}}(x)$  of the CGUE as an integral over the constraints  $\{B_q\}$ . We can interchange the roles of the  $\mathcal{P}$ -space and the  $\mathcal{Q}$ -space and study the distribution function  $P_{\mathcal{Q}}$  using the set  $\{B_p\}$  in  $\mathcal{P}$ -space as constraints. The probability distribution  $P_{\mathcal{Q}}(x)$  has the form of Eq. (15) with

$$F_{\mathcal{Q}}(x) = \int d[U] \left( \prod_p \delta(\langle B_p | UxU^\dagger \rangle) \right) \quad (43)$$

playing the role of  $F_{\mathcal{P}}(x)$ . The integrands of either function depend only on the eigenvalues  $b$  of the matrix  $B$  of the constraining ensemble. This suggests that it should be possible to express  $P_{\mathcal{P}}(x)$  as an integral over the complementary distribution function  $P_{\mathcal{Q}}(b)$ , and vice versa. In this Section we show that this is indeed the case and can be done via a generalization of Fourier's theorem.

We start from the identity

$$\prod_q \delta(\langle B_q | H \rangle) = \frac{1}{(2\pi)^{N_Q}} \int \prod_q dt_q \int \prod_p dt_p \exp \left( i \left\langle \sum_\sigma t_\sigma B_\sigma \right| H \right\rangle \right) \prod_p \delta(t_p) . \quad (44)$$

The matrices  $B_\sigma$  form a basis of  $\mathcal{V}$ . Therefore, the matrix  $B = \sum_\sigma t_\sigma B_\sigma$  is the general Hermitean matrix in  $N$  dimensions. By a suitable orthogonal transformation, we can replace the (general) basis matrices  $B_\sigma$  by the special basis set defined below Eq. (3). For the transformed integration variables  $\tilde{t}_\sigma$ , we have  $\prod_\sigma dt_\sigma = \prod_\sigma d\tilde{t}_\sigma$ . The product  $\prod_\sigma d\tilde{t}_\sigma$  is the product of the differentials of all matrix elements and, hence, equal to  $d[B]$ . The delta functions  $\delta(t_p)$  can be written identically as  $\delta(\langle B_p | B \rangle)$ . Thus,

$$\prod_q \delta(\langle B_q | H \rangle) = \frac{1}{(2\pi)^{N_Q}} \int d[B] \exp(i \langle B | H \rangle) \prod_p \delta(\langle B_p | B \rangle) . \quad (45)$$

The identity (45) relates the constraints in  $\mathcal{P}$ -space and in  $\mathcal{Q}$ -space via Fourier transformation. The Fourier integral is taken over Hermitean matrices in  $N$  dimensions.

We recall  $H = UxU^\dagger$ , write  $B$  as  $B = VbV^\dagger$ , and integrate the identity (45) over the unitary group  $U$ , using the definition (16) of  $F_{\mathcal{P}}(x)$  and the HCIZ integral. This yields

$$\begin{aligned} F_{\mathcal{P}}(x) &= \frac{1}{(2\pi)^{N_Q}} \int d[U] \int d[B] \exp \left( i \langle B | UxU^\dagger \rangle \right) \prod_p \delta(\langle B_p | B \rangle) \\ &= \frac{(2\pi)^{N(N-1)/2 - N_Q}}{\prod_{k=1}^N k!} \int d[U] \int d[V] \int d[b] \Delta^2(b) \exp \left( i \langle VbV^\dagger | UxU^\dagger \rangle \right) \\ &\quad \times \prod_p \delta(\langle B_p | VbV^\dagger \rangle) \\ &= \frac{(2\pi)^{N(N-1)/2 - N_Q}}{N! i^{N(N-1)/2}} \int d[b] \Delta(b) \frac{\det[\exp(ib_j x_k)]}{\Delta(x)} \\ &\quad \times \int d[V] \prod_p \delta(\langle B_p | VbV^\dagger \rangle) . \end{aligned} \quad (46)$$

Combining this result with the definition (43) of  $F_{\mathcal{Q}}(x)$ , we obtain the identity

$$\Delta(x) F_{\mathcal{P}}(x) = \frac{(2\pi)^{N(N-1)/2 - N_Q}}{N! i^{N(N-1)/2}} \int d[b] \det[\exp(ib_j x_l)] \Delta(b) F_{\mathcal{Q}}(b) . \quad (47)$$

This identity shows that  $\Delta(x)F_{\mathcal{P}}(x)$  is the Fourier transform of  $\Delta(b)F_{\mathcal{Q}}(b)$ . It is a generalization of Fourier's theorem to a class of symmetric functions of the variables  $x_1, x_2, \dots, x_N$ . It is the main result of this Section. We recall that  $F_{\mathcal{P}}(x)$  and  $F_{\mathcal{Q}}(b)$  are directly related to the joint probability functions  $P_{\mathcal{P}}(x)$  and  $P_{\mathcal{Q}}(b)$ , respectively, via Eq. (15). Thus, the complementary probability functions themselves are also obtained from each other by a generalized Fourier transform.

Writing the identity (47) for both  $F_{\mathcal{P}}(X)$  and  $F_{\mathcal{Q}}(b)$ , using the reality of  $F_{\mathcal{P}}$  and  $F_{\mathcal{Q}}$  in the latter relation, and inserting the result into the former, we find

$$\Delta(x)F_{\mathcal{P}}(x) = \frac{(2\pi)^{-N}}{(N!)^2} \int d[b] \int d[y] \det[\exp(ix_j b_l)] \det[\exp(-ib_j y_l)] \Delta(y)F_{\mathcal{P}}(y) . \quad (48)$$

Eq. (48) implies that

$$\frac{(2\pi)^{-N}}{(N!)^2} \int d[b] \det[\exp(ix_j b_l)] \det[\exp(-ib_j y_l)] = \prod_{l=1}^N \delta(x_l - y_l) \quad (49)$$

holds for a class of symmetric functions.

To illuminate the result (47), we discuss two examples. In the first example, the  $\mathcal{P}$ -space consists of all real symmetric matrices, and the  $\mathcal{Q}$ -space of all antisymmetric matrices with purely imaginary matrix elements. The joint distribution function of the GOE is known, and so is that of the (purely imaginary) antisymmetric Gaussian ensemble [17]. Both these functions are related via the generalized Fourier transform (47), and the explicit calculation is presented in the Appendix. In the second example, the  $\mathcal{P}$ -space consists of two block-diagonal matrices with equal dimensions. Thus, the eigenvalue distribution is that of a superposition of two GUEs. The complementary  $\mathcal{Q}$ -space is given by the chiral GUE used to describe universal phenomena of the QCD Dirac operator. The eigenvalue distribution for this ensemble is also known [18]. Both eigenvalue distributions are related to each other by the generalized Fourier transform.

## 7. Deformed Gaussian ensembles

So far, we have formulated the constraints in terms of delta functions (see Eq. (9)). It is tempting to consider a more general case where the matrices of the  $\mathcal{P}$ - and the  $\mathcal{Q}$ -space both have non-vanishing probability but differ in the widths of their Gaussian distribution functions. By changing the width of the distribution in  $\mathcal{Q}$ -space, we are able, for instance, to study the transition from the GUE to the CGUE. We refer to such ensembles as *deformed* Gaussian ensembles.

With  $\{B_\alpha\}$  an orthonormal basis in  $\mathcal{V}$ , we consider an ensemble of Hermitean matrices

$$H = \sum_p s_p B_p + \sum_q s_q B_q , \quad (50)$$

where  $s_p$  and  $s_q$  are real independent Gaussian variables with zero mean and variances

$$\overline{s_p^2} = \frac{1}{\lambda_P}, \quad \overline{s_q^2} = \frac{1}{\lambda_Q}. \quad (51)$$

Here and in what follows, we indicate ensemble averages by a bar over the quantity of interest. We note that for  $\lambda_P = 1$  and  $\lambda_Q = \infty$ , we recover the constrained ensemble (9), while for  $\lambda_P = 1 = \lambda_Q$ , we recover the GUE. We note that the deformed Gaussian ensemble (50) is not unitarily invariant unless  $\lambda_P = \lambda_Q$ . In calculating the spectral distribution function, we do not proceed as in the main part of Section 2, where we constructed first the unitarily invariant ensemble. Rather, we use the shortcut described below Eq. (16) and integrate directly over the unitary group. This procedure automatically generates the deformed Gaussian unitary ensemble (DGUE).

We make use of  $s_\sigma = \langle H|B_\sigma\rangle$ ,  $d[H] = \prod_\sigma ds_\sigma$ , and start from the probability density formula

$$W_{\mathcal{PQ}}(H, \lambda_P, \lambda_Q) d[H] = \left(\frac{\lambda_P}{2\pi}\right)^{\frac{N_P}{2}} \left(\frac{\lambda_Q}{2\pi}\right)^{\frac{N_Q}{2}} \exp\left(-\frac{\lambda_P}{2} \sum_p \langle H|B_p\rangle^2 - \frac{\lambda_Q}{2} \sum_q \langle H|B_q\rangle^2\right) d[H]. \quad (52)$$

We employ the completeness and orthonormality of  $\{B_\sigma\}$ , substitute

$$\lambda_P \sum_p \langle H|B_p\rangle^2 + \lambda_Q \sum_q \langle H|B_q\rangle^2 = \lambda_P \langle H|H\rangle + (\lambda_Q - \lambda_P) \sum_q \langle H|B_q\rangle^2, \quad (53)$$

express  $H$  in the diagonalized form,  $H = UxU^\dagger$ , and write  $d[H]$  as in Eq. (13). Inspecting the resulting formula, we find that the distribution function  $P_{\mathcal{PQ}}(x, \lambda_P, \lambda_Q)$  of the eigenvalues of the DGUE is given by the integral over the unitary group in  $N$  dimensions

$$P_{\mathcal{PQ}}(x, \lambda_P, \lambda_Q) = \left(\frac{\lambda_P}{2\pi}\right)^{\frac{N_P}{2}} \left(\frac{\lambda_Q}{2\pi}\right)^{\frac{N_Q}{2}} \frac{(2\pi)^{N(N-1)/2}}{\prod_{k=1}^N k!} \exp\left(-\frac{\lambda_P}{2} \langle x|x\rangle\right) \Delta^2(x) \\ \times \int d[U] \exp\left(-\frac{\lambda_Q - \lambda_P}{2} \sum_q \langle UxU^\dagger|B_q\rangle^2\right). \quad (54)$$

We use the Hubbard-Stratonovich transformation

$$\exp\left(-\frac{\lambda_Q - \lambda_P}{2} \langle UxU^\dagger|B_q\rangle^2\right) = \sqrt{\frac{\lambda}{2\pi}} \int dt_q \exp\left(it_q \langle UxU^\dagger|B_q\rangle - \frac{\lambda}{2} t_q^2\right) \quad (55)$$

where

$$\lambda = \frac{1}{\lambda_Q - \lambda_P} \quad (56)$$

and integrate over the unitary group with the help of the HCIZ formula (19). We obtain

$$P_{\mathcal{PQ}}(x, \lambda_P, \lambda_Q) = \left(\frac{\lambda_P}{2\pi}\right)^{\frac{N_P}{2}} \left(\frac{\lambda_Q}{2\pi}\right)^{\frac{N_Q}{2}} \frac{(2\pi)^{N(N-1)/2}}{N! i^{N(N-1)/2}} \exp\left(-\frac{\lambda_P}{2} \langle x|x\rangle\right) \Delta^2(x) \\ \times \left(\frac{\lambda}{2\pi}\right)^{\frac{N_Q}{2}} \int \prod_q dt_q \exp\left(-\frac{\lambda}{2} t_q^2\right) \frac{\det[\exp(ix_k b_l)]}{\Delta(x)\Delta(b)}. \quad (57)$$



The  $\{b_l\}$  are the eigenvalues of the matrix  $B = \sum_q t_q B_q$ . We note that for  $\lambda_Q \rightarrow \lambda_P = 1$ , we have  $\lambda \rightarrow \infty$ , the Gaussian factors turn into delta functions, the integral over the  $\{t_q\}$  becomes a constant, and  $P_{\mathcal{PQ}}(x, \lambda_P, \lambda_Q)$  turns into the GUE distribution. On the other hand, for  $\lambda_P = 1$  and  $\lambda_Q \rightarrow \infty$ , we have  $\lambda \rightarrow 0$ , the Gaussian cutoff becomes irrelevant, and the right-hand side of Eq. (57) approaches that of Eq. (22). Thus, the DGUE correctly interpolates between the GUE and the CGUE.

We recall the discussion in Section 4 and observe that the integrals on the right-hand side of Eq. (57) converge for all positive values of  $\lambda$ . This shows that in the DGUE,  $P_{\mathcal{PQ}}(x, \lambda_P, \lambda_Q)$  vanishes whenever two eigenvalues  $x_k, x_l$  coincide no matter how large  $\lambda_Q$ , and level repulsion prevails. If level repulsion disappears, it does so abruptly at  $\lambda_Q = \infty$ . On physical grounds we expect, of course, that as  $\lambda_Q$  increases, level repulsion affects ever smaller distances between levels.

The last term on the right-hand side of Eq. (57) can be reinterpreted as the ensemble average of  $\det[\exp(ix_j b_k)]/(\Delta(x)\Delta(b))$  over the  $\mathcal{Q}$ -space ensemble of matrices  $B = \sum_q t_q B_q$  with Gaussian-distributed  $t_q$  with the variances  $\bar{t}_q^2 = 1/\lambda$ . We have

$$\left(\frac{\lambda}{2\pi}\right)^{\frac{N_Q}{2}} \int \prod_q dt_q \exp\left(-\frac{\lambda}{2} t_q^2\right) \frac{\det[\exp(ix_k b_l)]}{\Delta(x)\Delta(b)} = \int d[b] P_Q(b, \lambda) \frac{\det[\exp(ix_k b_l)]}{\Delta(x)\Delta(b)} \quad (58)$$

where (cf. Eq. (15))

$$P_Q(b, \lambda) = \frac{(2\pi)^{N(N-1)/2}}{\prod_{k=1}^N k!} \left(\frac{\lambda}{2\pi}\right)^{\frac{N_Q}{2}} \exp\left(-\frac{\lambda}{2} \langle b|b \rangle\right) \Delta^2(b) \int d[U] \prod_p \delta(\langle B_p | U b U^\dagger \rangle) \quad (59)$$

denotes the distribution function of eigenvalues  $b_k$  of this ensemble. This shows that  $P_{\mathcal{PQ}}(x, \lambda_P, \lambda_Q)$  can be expressed entirely through the rescaled eigenvalue distribution  $P_Q(b, \lambda)$ :

$$P_{\mathcal{PQ}}(x, \lambda_P, \lambda_Q) = \left(\frac{\lambda_P}{2\pi}\right)^{\frac{N_P}{2}} \left(\frac{\lambda_Q}{2\pi}\right)^{\frac{N_Q}{2}} \frac{(2\pi)^{N(N-1)/2}}{N! i^{N(N-1)/2}} \exp\left(-\frac{\lambda_P}{2} \langle x|x \rangle\right) \Delta^2(x) \\ \times \int d[b] P_Q(b, \lambda) \frac{\det[\exp(ix_k b_l)]}{\Delta(x)\Delta(b)}. \quad (60)$$

Throughout the derivations in this Section we may, of course, exchange the roles of the  $\mathcal{P}$ -space and of the  $\mathcal{Q}$ -space. The resulting equations are obtained by substituting  $P \leftrightarrow Q$  except that the term in the exponent on the left-hand side of Eq. (55) is now positive. This causes the imaginary unit  $i$  to disappear in that and all following equations. We note that the generalized Fourier transformation of Section 6 can be extended to the DGUE. We also note that the transition from GUE to GOE, or from GUE to the antisymmetric Gaussian ensemble, can be described in the framework of the DGUE. In this case, one recovers the result given in Ref. [19].

## 8. Summary

Introducing sets of constraints, we have defined new classes of random-matrix ensembles, the constrained Gaussian unitary (CGUE), and the deformed Gaussian unitary (DGUE)

ensembles. The CGUEs consist of Hermitean random matrices where the constraints set certain linear combinations of matrix elements to zero. The DGUEs interpolate between the GUE and the corresponding CGUE. Using the Harish–Chandra Itzykson Zuber integral for the CGUE, we have found a sufficient condition for GUE-type level repulsion to persist in the presence of constraints. This condition depends only on the number of constraints and on the internal symmetries of the constraining matrices and can be formulated as a simple inequality. We have derived a generalized Fourier theorem which relates the spectral properties of the constraining ensembles with those of the constrained ones. We have shown that in the DGUEs, level repulsion always prevails at sufficiently short distances. It is only in the limit of strictly enforced constraints that level repulsion may be lifted. As a result, we find that GUE-type level repulsion is remarkably robust. The extension of this approach to the orthogonal and the symplectic cases is hampered by the fact that the analogues of the Harish–Chandra Itzykson Zuber integral formula are not known in closed form. In these cases, we found a sufficient condition for the canonical level repulsion only for certain classes of constraints.

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## Appendix: Relation between the GOE and the antisymmetric Gaussian ensemble

The Gaussian antisymmetric ensemble and the GOE are complementary ensembles. In this appendix, we show that the eigenvalue distributions of GOE and the antisymmetric Gaussian ensemble are related to each other by the generalized Fourier transform (47). For simplicity, we consider matrix ensembles with matrices of even dimension  $N = 2m$ . We start by considering the GOE basis matrices as the constraints and use the probability distribution of the antisymmetric Gaussian ensemble as given in Ref. [17]

$$P_{\mathcal{P}}(b) \propto \mathcal{S} \exp \left( -\frac{1}{2} \langle b|b \rangle \right) \prod_{1 \leq j < k \leq m} (b_j^2 - b_k^2)^2 \prod_{l=1}^m \delta(b_l + b_{m+l}) . \quad (\text{A.1})$$

We have omitted an overall normalization constant. The symbol  $\mathcal{S}$  denotes symmetrization with respect to all eigenvalues. According to Eq. (15), one thus finds

$$\begin{aligned} \Delta(b) F_{\mathcal{P}}(b) &\propto \mathcal{A} \prod_{1 \leq j < k \leq m} (b_j^2 - b_k^2)^2 \prod_{l=1}^m \delta(b_l + b_{m+l}) / \Delta(b) \\ &\propto \mathcal{A} \prod_{k=1}^m \frac{\delta(b_k + b_{m+k})}{b_k} . \end{aligned} \quad (\text{A.2})$$

The symbol  $\mathcal{A}$  denotes antisymmetrization with respect to all eigenvalues. We have used the identity

$$\prod_{1 \leq k < l \leq m} (x_k^2 - x_l^2)^2 \prod_{j=1}^m \delta(x_j + x_{j+m}) = \Delta(x) \prod_{j=1}^m \frac{\delta(x_j + x_{j+m})}{x_j - x_{j+m}}, \quad (\text{A.3})$$

which can be proved by expanding the products and exploiting the  $\delta$ -functions. We insert the last relation (A.2) into the Fourier relation (47) and find

$$\Delta(x) F_{\mathcal{Q}}(x) \propto \int \prod_{l=1}^m \frac{db_l}{b_l} \det [\exp (ib_j x_k)] . \quad (\text{A.4})$$

We have performed one-half of the integrations by means of the  $\delta$ -functions, and  $b_{m+j} = -b_j$  thus holds in the exponential. We have dropped the antisymmetrizer  $\mathcal{A}$  since the determinant is antisymmetric anyway. We use the Laplace expansion of the determinant and obtain terms of the form

$$\prod_{k=1}^m \int \frac{db_k}{b_k} \exp (ib_k (x_{j_k} - x_{j_{m+k}})) = (i\pi)^m \prod_{k=1}^m \text{sign}(x_{j_k} - x_{j_{m+k}}) . \quad (\text{A.5})$$

Summing up all terms, one finds

$$\Delta(x) F_{\mathcal{Q}}(x) \propto \mathcal{A} \prod_{k=1}^m \text{sign}(x_{j_k} - x_{j_{m+k}}) . \quad (\text{A.6})$$

This expression is totally antisymmetric under permutations of any two eigenvalues and is a homogeneous function of degree zero. Thus,

$$\Delta(x) F_{\mathcal{Q}}(x) \propto \text{sign}(\Delta(x)) , \quad (\text{A.7})$$

and this yields the GOE distribution once Eq. (15) is employed.

We can also employ the Fourier transform to go from the GOE to the antisymmetric Gaussian ensemble. For this purpose, we insert the GOE expression (A.7) into the Fourier relation (47) and find

$$\Delta(x) F_{\mathcal{P}}(x) \propto \int d[b] \text{sign}(\Delta(x)) \det [\exp (ib_j x_k)] . \quad (\text{A.8})$$

This integral can be performed by using, for instance, the results presented in Sect. 3 of Ref. [19]. We find

$$\Delta(x) F_{\mathcal{P}}(x) \propto \text{Pf}[a_{jk}]_{j,k=1 \dots 2m} . \quad (\text{A.9})$$

Here,

$$\begin{aligned} a_{jk} &\equiv \int_{u < v} du dv \left( e^{ix_j u} e^{ix_k v} - e^{ix_k u} e^{ix_j v} \right) \\ &= 8\pi i (x_k - x_j)^{-1} \delta(x_j + x_k) , \end{aligned} \quad (\text{A.10})$$

and Pf denotes the Pfaffian

$$\text{Pf}[a_{jk}] = \sum_P (-1)^P a_{j_1 j_2} a_{j_3 j_4} \dots a_{j_{2m-1} j_{2m}} , \quad (\text{A.11})$$

with  $P$  running over all  $(2m)!/(2^m m!)$  permutations  $(j_1, \dots, j_{2m})$  of  $(1, 2, \dots, 2m)$  with restrictions  $j_1 < j_2, j_3 < j_4, \dots, j_{2m-1} < j_{2m}$  and  $i_1 < i_3 < \dots, i_{2m-1}$ .

The expression (A.9) is thus completely antisymmetric under permutations of any two of its arguments. The probability density we seek is essentially this expression, multiplied by the Vandermonde determinant  $\Delta(x)$ , and is thus completely symmetric under permutations of its arguments. We may therefore focus on just one term of the Pfaffian and write

$$P_{\mathcal{P}}(x) \propto \mathcal{S} \exp\left(-\frac{1}{2}\langle x|x\rangle\right) \Delta(x) \prod_{k=1}^m (x_k - x_{k+m})^{-1} \delta(x_k + x_{k+m}) . \quad (\text{A.12})$$

Employing the  $\delta$ -functions and the identity (A.3), we recover Eq. (A.1).

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